A counterexample to the simple loop conjecture for $\mathrm{PSL}(2,\mathbb{R})$

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Abstract

In this note, we give an explicit counterexample to the simple loop conjecture for representations of surface groups into $\mathrm{PSL}(2,\mathbb{R})$. Specifically, we use a construction of DeBlois and Kent to show that for any orientable surface with negative Euler characteristic and genus at least 1, there are uncountably many non-conjugate, non-injective homomorphisms of its fundamental group into $\mathrm{PSL}(2,\mathbb{R})$ that kill no simple closed curve (nor any power of a simple closed curve). This result is not new – work of Louder and Calegari for representations of surface groups into $\mathrm{SL}(2,\mathbb{C})$ applies to the $\mathrm{PSL}(2,\mathbb{R})$ case, but our approach here is explicit and elementary.

1 Introduction

The simple loop conjecture, proved by Gabai in [Gabai 1985], states that any non-injective homomorphism from a closed surface group to another closed surface group has an element represented by a simple closed curve in the kernel. It has been conjectured that the result still holds if the target is replaced by the fundamental group of an orientable 3-manifold (see Kirby's problem list in [Kirby 1993]). Although special cases have been proved (e.g. [Hass 1987, Rubinstein and Wang 1998]), the general hyperbolic case is still open.

Minsky [2000] asked whether the conjuecture holds if the target group is instead $\mathrm{SL}(2,\mathbb{C})$. This was answered in the negative by Cooper and Manning with the following theorem.

Theorem 1.1. [Cooper and Manning 2011]. Let Σ be a closed orientable surface of genus $g \geq 4$. Then there is a homomorphism $\rho : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{C})$ such that

1. ρ is not injective
2. If $\rho(\alpha) = \pm I$, then α is not represented by a simple closed curve
3. If $\rho(\alpha)$ has finite order, then $\rho(\alpha) = I$

The third condition implies in particular that no *power* of a simple closed curve lies in the kernel.

Inspired by this, we ask whether a similar result holds for $\mathrm{PSL}(2,\mathbb{R})$, this being an intermediate case between Gabai's result for surface groups and Cooper and Manning's for $\mathrm{SL}(2,\mathbb{C})$. Techniques of Cooper and Manning's proof do not seem to carry over directly to the $\mathrm{PSL}(2,\mathbb{R})$ case – their work involves both a dimension count on the $\mathrm{SL}(2,\mathbb{C})$ character variety and a proof showing that a specific subvariety is irreducible and smooth on a dense subset, and complex varieties and their real points generally behave quite differently. However, we will show here with different methods that an analog to Theorem 1.1 does hold for $\mathrm{PSL}(2,\mathbb{R})$.

While this note was in progress, we learned of work of Louder and Calegari (independently in [Louder 2011] and [Calegari 2011]) that can also be applied to answer our question in the affirmative. Louder shows the simple loop conjecture is false for representations into limit groups, and Calegari gives a practical way of verifying no simple closed curves lie in the kernel of a non-injective representation using stable commutator length and the Gromov norm.

The difference here is that our construction is entirely elementary. We use an explicit representation found in [DeBlois and Kent 2006] (which uses work from [Goldman 1988] and [Shalen 1979]), and we verify by elementary means that this representation is non injective and kills no simple closed curve. Our end result parallels that of Cooper and Manning but also include surfaces with boundary and all genera at least 1:

Theorem 1.2. Let Σ be an orientable surface of negative Euler characteristic and of genus $g \geq 1$, possibly with boundary. Then there is a homomorphism $\rho : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbb{R})$ such that

1. ρ is not injective
2. If $\rho(\alpha) = \pm I$, then α is not represented by a simple closed curve
3. In fact, if α is represented by a simple closed curve, then $\rho(\alpha^k) \neq I$ for any $k \in \mathbb{Z}$.

Moreover, there are uncountably many non-conjugate representations satisfying 1. through 3.

In the case of a non-orientable surface, the appropriate target group is $\mathrm{PGL}(2, \mathbb{R})$, as the fundamental group of a non-orientable hyperbolic surface can be represented as a lattice in $\mathrm{PGL}(2, \mathbb{R})$. This again gives an intermediate case between the simple loop conjecture for representations into surface groups and into $\mathrm{PSL}(2, \mathbb{C})$. We have the following direct generalization of Theorem 1.2, with essentially the same proof.

Theorem 1.3. Let Σ be a non-orientable surface of negative Euler characteristic and of non-orientable genus $g \geq 2$, not the punctured Klein bottle nor the closed non-orientable genus 3 surface. Then there are uncountably many representations $\rho : \pi_1(\Sigma) \to \mathrm{PGL}(2, \mathbb{R})$ satisfying conditions 1. through 3. of Theorem 1.2.

See section 3 for a comment on the exceptional cases of the punctured Klein bottle and closed, non-orientable genus 3 surface.

2 Proof of Theorem 1.2

We describe a family of (non-injective) representations constructed in [DeBlois and Kent 2006] based on a construction of [Goldman 1988]. We will then show that this family contains infinitely many non-conjugate representations with no simple closed curve in the kernel.

Let Σ be an orientable surface of genus $g \geq 1$ and negative Euler characteristic, possibly with boundary. Assume for the moment that Σ is not the once-punctured torus – Theorem 1.2 for this case will follow easily later on.

Let $c \subset \Sigma$ be a simple closed curve separating Σ into a genus 1 subsurface with single boundary component c, and a genus $(g-1)$ subsurface with one or more boundary components. Let Σ_A denote the genus $(g-1)$ subsurface and Σ_B the genus 1 subsurface. Finally, we let $A = \pi_1(\Sigma_A)$ and $B = \pi_1(\Sigma_B)$, so that $\pi_1(\Sigma) = A *_C B$, where C is the infinite cyclic subgroup generated by the element $[c]$ represented by the curve c. We assume that the basepoint for $\pi_1(\Sigma)$ lies on c.

Let $x \in B$ and $y \in B$ be generators such that $B = \langle x, y \rangle$, and that the curve c represents the commutator $[x, y]$. See Figure 1.

Fix α and β in $\mathbb{R} \setminus \{0, \pm 1\}$, and following [DeBlois and Kent 2006] define $\phi_B : B \to \mathrm{SL}(2, \mathbb{R})$ by

$$\phi_B(x) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

$$\phi_B(y) = \begin{pmatrix} \beta & 1 \\ 0 & \beta^{-1} \end{pmatrix}$$

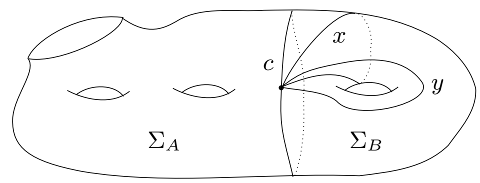

Figure 1: Decomposition of Σ and curves representing generators x and y for B

We have then
$$\phi_B([x,y]) = \begin{pmatrix} 1 & \beta(\alpha^2-1) \\ 0 & 1 \end{pmatrix}$$
so that $\phi_B([x,y])$ is invariant under conjugation by the matrix $\lambda_t := \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$.

After projecting these matrices to $\mathrm{PSL}(2,\mathbb{R})$ we have a representation $B \to \mathrm{PSL}(2,\mathbb{R})$ which is upper triangular, hence solvable, and therefore non-injective. Abusing notation, we let ϕ_B denote this representation.

Now let $\phi_A : A \to \mathrm{PSL}(2,\mathbb{R})$ be Fuchsian and such that the image of the boundary curve c under ϕ_A agrees with $\phi_B([x,y])$. That such a representation exists is standard – Σ_A has negative Euler characteristic and therefore admits a complete hyperbolic structure. The image of $[c]$ under the corresponding Fuchsian representation is a parabolic element of $\mathrm{PSL}(2,\mathbb{R})$, so after conjugation we may assume that it is equal to $\phi_B([x,y])$, since $\beta(\alpha^2-1) \neq 0$.

Finally, we combine ϕ_A with conjugates of ϕ_B to get a one-parameter family of representations $\phi_t : \pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{R})$ as follows. For $t \in \mathbb{R}$ and $g \in \pi_1(\Sigma) = A *_C B$, let
$$\phi_t(g) = \begin{cases} \phi_A(g) & \text{if } \; g \in A \\ \lambda_t \circ \phi_B(g) \circ (\lambda_t)^{-1} & \text{if } \; g \in B \end{cases}$$
This representation is well defined because $\phi_B([x,y]) = \phi_A([x,y])$ and is invariant under conjugation by λ_t.

Our next goal is to show that for appropriate choice of α, β, and t, the representation ϕ_t satisfies the criteria in Theorem 1.2. The main difficulty will be checking that no element representing a simple closed curve is of finite order. To do so, we employ a stronger form of Lemma 2 from [DeBlois and Kent 2006]. This trick originally comes from the proof of Proposition 1.3 in [Shalen 1979].

Lemma 2.1. Suppose $w \in A *_C B$ is a word of the form $w = a_1 b_1 a_2 b_2 ... a_l b_l$ with $a_i \in A$ and $b_i \in B$ for $1 \leq i \leq l$. Assume that for each i, the matrix $\phi_0(a_i)$ has a nonzero 2,1 entry and $\phi_0(b_i)$ is hyperbolic. If t is transcendental over the entry field of $\phi_0(A *_C B)$, then $\phi_t(w)$ is not finite order.

By *entry field* of a group Γ of matrices, we mean the field generated over $\mathbb{Q}$ by the collection of all entries of matrices in Γ.

Remark 2.2. Lemma 2 of [DeBlois and Kent 2006] is a proof that $\phi_t(w)$ is not the *identity*, under the assumptions of Lemma 2.1. We use some of their work in our proof.

Proof of Lemma 2.1. DeBlois and Kent show by a straightforward induction (we leave it as an exercise) that under the hypotheses of Lemma 2.1, the entries of $\phi_t(w)$ are polynomials in t such that the degree of the 2,2 entry is l, the degree of the 1,2 entry is at most l, and the other entries have degree at most $l-1$. Now suppose that $\phi_t(w)$ is finite order. Then it is conjugate to a matrix of the form $\left(\begin{smallmatrix} u & v \\ -v & u \end{smallmatrix}\right)$. where $u = \cos(\theta)$ and $v = \sin(\theta)$ for θ a rational multiple of π. In particular, it follows from the deMoivre formula for sine and cosine that u and v are algebraic.

Now suppose that the matrix conjugating $\phi_t(w)$ to $\left(\begin{smallmatrix} u & v \\ -v & u \end{smallmatrix}\right)$ has entries a_{ij}. Then we have

$$\phi_t(w) = \begin{pmatrix} u - (a_{12}a_{22} - a_{11}a_{21})v & (a_{12}^2 a_{11}^2)v \\ -(a_{22}^2 a_{21}^2)v & u + (a_{12}a_{22} + a_{11}a_{21})v \end{pmatrix}$$

Looking at the 2,2 entry we see that $a_{12}a_{22} + a_{11}a_{21}$ must be a polynomial in t of degree l. But this means that the 1,1 entry is also a polynomial in t of degree l, contradicting Deblois and Kent's calculation. This proves the lemma. □

To complete our construction, choose any $t \in \mathbb{R}$ that is transcendental over the entry field of $\phi_0(A *_C B)$. We want to show that no power of an element representing a simple closed curve lies in the kernel of ϕ_t. To this end, consider any word w in $A *_C B$ that has a simple closed curve as a representative. There are three cases to check.

Case i) w is a word in A alone
In this case $\phi_t(w)$ is not finite order, since $\phi_t(A)$ is Fuchsian and therefore injective.

Case ii) w is a word in B alone
Theorem 5.1 of [Birman and Series 1984] states that w can be represented by a simple closed curve only if it has one of the following forms after cyclic reduction:

1. $w = x^{\pm 1}$ or $w = y^{\pm 1}$
2. $w = [x^{\pm 1}, y^{\pm 1}]$
3. Up to replacing x with x^{-1}, y with y^{-1} and interchanging x and y, there is some $n \in \mathbb{Z}^+$ such that $w = x^{n_1} y x^{n_2} y ... x^{n_s} y$ where $n_i \in \{n, n+1\}$.

The heuristic for case 3 of the Birman-Series theorem is shown in Figure 2 – if w is represented by a simple closed curve and terminates with $x^{n_s} y$, this forces the rest of the curve representing w to wind around the punctured torus in a set pattern. The figure shows the behavior for $n_s = 4$.

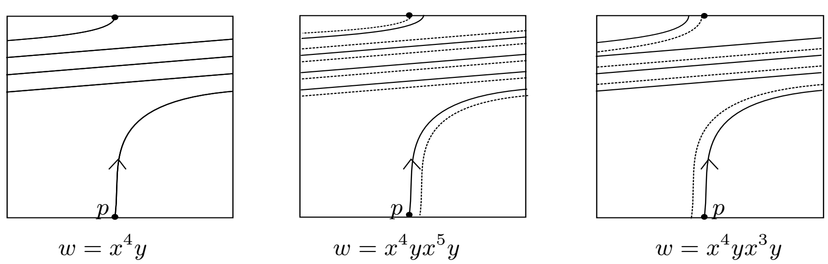

Figure 2: Simple closed curves on the once punctured torus
Assume the puncture is at the vertex, x is represented by a horizontal loop oriented from left to right, and y is a vertical loop oriented from bottom to top.

By construction, no word of type 1, 2 or 3 above is finite order provided that $\alpha^s \beta^k \neq 1$ for any integers s and k other than zero – indeed, we only need to check words of type 3, and these necessarily have trace equal to $\alpha^s \beta^k + \alpha^{-s} \beta^{-k}$ for some $s, k \neq 0$. Since cyclic reduction corresponds to conjugation, no word in B has image a finite order element.

Note also that, in particular, under the condition that $\alpha^s \beta^k \neq 1$ for $s, k \neq 0$, all type 3 words are hyperbolic. We will use this fact again later on.

Case iii) general case
If w is a word with both A and B, we claim that it can be written in a form where Lemma 2.1 applies. To write it this way, take a simple curve γ on Σ that represents w and has a minimal number of (geometric) intersections with c. We can write γ as a concatenation of simple arcs $\gamma = \gamma_1\delta_1\gamma_2\delta_2...\gamma_n\delta_n$ with $\gamma_i \subset \Sigma_A$ and $\delta_i \subset \Sigma_B$. Since we chose γ to have a minimal number of intersections with c, no arc γ_i (or δ_i) is homotopic in Σ_A (respectively in Σ_B) to a segment of c – if it were, we could apply an isotopy of Σ supported in a neighborhood of the disc bounded by the arc and the segment of c to push the arc across c and reduce the total number of intersections.

Now choose a proper segment c' of c that contains both the basepoint p and all endpoints of all γ_i and δ_i, and close each of the arcs γ_i and δ_i into a simple loop by attaching a segment of c'. If $a_i \in A$ and $b_i \in B$ are represented by the loops γ_i and δ_i, then $a_1b_1a_2b_2...a_nb_n = w$ in $\pi_1(\Sigma)$.

Since no arc γ_i or δ_i was homotopic to a segment of c, no a_i or b_i is represented by a power of $[c]$ in $\pi_1(\Sigma)$. We claim that in this case $a_1b_1a_2b_2...a_nb_n$ satisfies the hypotheses of Lemma 2.1. Indeed, since ϕ_A is Fuchsian, the only elements with a non-zero $2,1$ entry are powers of $[c]$, and the Birman-Series classification of simple closed curves on Σ_b implies that the only simple closed curves which are *not* hyperbolic represent $[c]$ or $[c]^{-1}$.

It remains only to remark that the representation ϕ_t is non-injective and that, by choosing appropriate parameters, we can produce uncountably many nonconjugate representations. Non-injectivity follows immediately since $\phi_t(B)$ is solvable so the restriction of ϕ_t to B is non-injective. Now for any fixed α and β (satisfying the requirement that $\alpha^s\beta^k \neq 1$ for all integers s, k), varying t among transcendentals over the entry field of $\phi_0(A *_C B)$ produces uncountably many non-conjugate representations that are all non-injective, but have no power of a simple closed curve in the kernel. This concludes the proof of Theorem 1.2, assuming that the surface was not the punctured torus.

The punctured torus case is now immediate: any representation of the form of ϕ_B where $\alpha^s\beta^k \neq 1$ for any integers s and k is non-injective and our work above shows that no element represented by a simple closed curve has finite order. Fixing α and varying β produces uncountably many non-conjugate representations.

3 Non-orientable surfaces

Recall that the *genus* of a non-orientable surface Σ is defined to be the number of $\mathbb{R}\mathrm{P}^2$ summands in a decomposition of the surface as $\Sigma = \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2 \# ... \# \mathbb{R}\mathrm{P}^2$. A closed, non-orientable genus g surface has Euler characteristic $\chi = 2 - g$.

Let Σ be a non-orientable surface of negative Euler characteristic and non-orientable genus $g \geq 2$, not the punctured Klein bottle nor the closed non-orientable genus 3 surface. The same strategy as in the orientable case can then be used to produce uncountably many non injective representations $\pi_1(\Sigma) \to \mathrm{PGL}(2, \mathbb{R})$ such that no power of a simple closed curve lies in the kernel. In detail, our assumptions on Σ imply that we may decompose Σ along a (2-sided) curve c into a genus 1 orientable surface Σ_B with one boundary component and a non-orientable surface Σ_A of negative Euler characteristic.

We define ϕ_B exactly as in the orientable case, but now consider the matrices as elements of $\mathrm{PGL}(2, \mathbb{R})$ rather than $\mathrm{PSL}(2, \mathbb{R})$. We let $\phi_A : \pi_1(\Sigma_A) \to \mathrm{PGL}(2, \mathbb{R})$ be a discrete, faithful representation such that $\phi_A([c]) = \phi_B([c])$. As in the case of the orientable surface, we may take this to be a representation corresponding to a complete hyperbolic structure on Σ. Define $\phi_t : \pi_1(\Sigma) \to \mathrm{PGL}(2, \mathbb{R})$ by "gluing together" ϕ_A with a conjugate of ϕ_B by λ_t exactly as in the orientable case. The proof now carries through verbatim; for none of the topological arguments that we used required orientability of Σ_A. We also reassure the reader (who may be unfamiliar with lattices in $\mathrm{PGL}(2, \mathbb{R})$) that powers of $\phi_A([c])$ are indeed the only elements of the image of ϕ_A with 0 as the 2,1-entry.

This strategy does not cover the case of the punctured Klein bottle, which cannot be decomposed with a T^2 summand, nor the closed non-orientable genus 3 surface, which decomposes as $T^2 \# \mathbb{R}\mathrm{P}^2$. It would be interesting to try to cover this case in a manner analogous to the punctured torus case of Theorem 1.2 by providing a classification of simple closed curves on these surfaces. Indeed (as the referee has pointed out) the punctured Klein bottle case is not too difficult. The closed, non-orientable genus 3 surface case appears to be more challenging.

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